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Poisson Integrals and Singular Weights

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An earlier investigation into power series coefficients of functions in $H^2(\Pi_+)$ [2] brought into consideration a class of functions which belong to $L^2(\mathbf{R})$ with respect to a weight possessing a singularity at the origin as well as the Poisson integrals of those functions. Here we investigate the space of those Poisson integrals as a subspace of all harmonic functions in the upper halfplane with L^2 boundary values on the real line.

The weight originally considered behaved like $|t|^{-1}$ near the origin; here we consider a more general set of weights, those whose behavior near zero is of the form $|t|^{-\alpha}$ for $\alpha \geqslant 0$. In particular we consider the space

$$C_\alpha = \left\{ f \colon \int_{-\infty}^{\infty} |f(t)|^2 \left(\frac{t^2+1}{t^2} \right)^{\alpha/2} dt = \|f\|_{C_\alpha}^2 < \infty \right\}.$$

Clearly C_α is a proper subspace of $L^2(\mathbf{R})$ and hence we can look at the Poisson integrals of these functions defining

$$D_\alpha = \{ F = P[f] \colon f \in C_\alpha \}.$$

The goal here is to characterize D_α in terms of the growth conditions of these harmonic functions. We seek a correspondence between D_α and spaces of form

$$\tilde{D}_\alpha = \left\{ F \text{ harmonic in } \mathbf{R}^2_+ : \sup_{y>0} \int_{-\infty}^{\infty} |F(x, y)|^2 \, W_\alpha(x, y) \, dx = \|F\|_{\tilde{D}_\alpha}^2 < \infty \right\}.$$

The problem is to determine W_α so that $D_\alpha = \tilde{D}_\alpha$.

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Some observations are immediate. If $F \in D_\alpha$ then, since $C_\alpha \subset L^2$,

$$\sup_{y>0} \int_{-\infty}^{\infty} |F(x, y)|^2 \, dx < \infty;$$

however, this does not guarantee inclusion in $\tilde{D}_\alpha$. Conversely, if $W_\alpha \geqslant K > 0$, then $F \in \tilde{D}_\alpha$ implies the existence of an $f \in L^2$ such that $F = P[f]$. But again there is no a priori reason to suspect that $f \in C_\alpha$.

The results we demonstrate complete these observations and are of the form:

THEOREM. *If $W_\alpha(x, y) = g_\alpha(x, y)$, then $D_\alpha = \tilde{D}_\alpha$ for $\alpha \in (\alpha_1, \alpha_2)$.*

The first surmise might be that $W_\alpha(x, y) = w_\alpha(x) = ((x^2+1)/x^2)^{\alpha/2}$. A simple example, letting f be the characteristic function of the interval $[1, 2]$ with $\alpha = 1$, refutes this conjecture [2]. However, modeling $W_\alpha(x, y)$ after $w_\alpha(t)$ led us to the first of our results.

THEOREM 1. *If $W_\alpha(x, y) = ((x^2+y^2+1)/(x^2+y^2))^{\alpha/2}$, then $D_\alpha = \tilde{D}_\alpha$ for $0 \leqslant \alpha < 3$. Moreover, if $F \in D_\alpha$, where $F = P[f]$, then $\|f\|_{C_\alpha} \leqslant \|F\|_{\tilde{D}_\alpha} \leqslant C\,\|f\|_{C_\alpha}$.*

This theorem and those to follow rely heavily upon the properties of the Poisson integral and the Hardy–Littlewood maximal function. In particular we utilize the boundedness of the maximal function as an operator from L^2 to L^2, the domination of the Poisson integral of an L^2 function by that function's maximal function (this dominance is uniform in y), and the almost everywhere convergence of the Poisson integral of an L^2 function to that function as $y \to 0$. These well-known results can be found in [1, 3, 4].

Proof. First we show $\tilde{D}_\alpha$ is contained in D_α. If $F \in \tilde{D}_\alpha$, then F is harmonic in $\mathbf{R}^2_+$ and has L^2 boundary values; hence there is an $f \in L^2(\mathbf{R})$ such that $F = P[f]$ and $\lim_{y \to 0} F(x, y) = f(x)$ a.e. The first assertion follows in exactly the same way as the analagous result for $F \in H^2(\Pi_+)$; see [5, Chap. 5]. Thus it only remains to show that $f \in C_\alpha$. Let

$$F_1(x, y) = F(x, y)\left(\frac{x^2+y^2+1}{x^2+y^2}\right)^{\alpha/2}.$$

Then

$$\lim_{y \to 0} |F_1(x, y)|^2 = |f(x)|^2 \left(\frac{x^2+1}{x^2}\right)^{\alpha/2} \quad \text{a.e.}$$

and by Fatou's lemma,

$$\begin{aligned}
\|f\|_{C_\alpha}^2 &= \int_{-\infty}^{\infty} |f(x)|^2 \left(\frac{x^2+1}{x^2}\right)^{\alpha/2} dx \\
&= \int_{-\infty}^{\infty} \lim_{y\to 0} |F_1(x, y)|^2\, dx \\
&\leqslant \varliminf_{y\to 0} \int_{-\infty}^{\infty} |F_1(x, y)|^2\, dx \\
&= \varliminf_{y\to 0} \int_{-\infty}^{\infty} |F(x, y)|^2 \left(\frac{x^2+y^2+1}{x^2+y^2}\right)^{\alpha/2} dx \\
&\leqslant \|F\|_{\tilde{D}_\alpha}^2 < \infty,
\end{aligned}$$

since $F \in \tilde{D}_\alpha$. Therefore $f \in C_\alpha$, and $F \in D_\alpha$ with $\|f\|_{C_\alpha} \leqslant \|F\|_{\tilde{D}_\alpha}$. Observe that the only property of $W_\alpha(x, y) = ((x^2+y^2+1)/(x^2+y^2))^{\alpha/2}$ needed is

$$\lim_{y\to 0} W_\alpha(x, y) = \left(\frac{x^2+1}{x^2}\right)^{\alpha/2}.$$

Next assume $F \in D_\alpha$.

$$\begin{aligned}
&\sup_{y>0} \int_{-\infty}^{\infty} |F(x, y)|^2\, W_\alpha(x, y)\, dx \\
&\qquad \leqslant \sup_{0<y\leqslant 1} \int_{-\infty}^{\infty} |F(x, y)|^2\, W_\alpha(x, y)\, dx + \sup_{y>1} \int_{-\infty}^{\infty} |F(x, y)|^2\, W_\alpha(x, y)\, dx.
\end{aligned}$$

However, if $y > 1$, then

$$W_\alpha(x, y) = \left(\frac{x^2+y^2+1}{x^2+y^2}\right)^{\alpha/2} = \left(1 + \frac{1}{x^2+y^2}\right)^{\alpha/2} \leqslant C$$

and

$$\begin{aligned}
\sup_{y>1} \int_{-\infty}^{\infty} |F(x, y)|^2\, W_\alpha(x, y)\, dx
&\leqslant \sup_{y>1} C \int_{-\infty}^{\infty} |F(x, y)|^2\, dx \\
&\leqslant \sup_{y>0} C \int_{-\infty}^{\infty} |F(x, y)|^2\, dx. \qquad (1)
\end{aligned}$$

Moreover, since C_α is a subspace of $L^2(\mathbf{R})$, F is the Poisson integral of an L^2 function and thus

$$\sup_{y>0} \int_{-\infty}^{\infty} |F(x, y)|^2\, dx \leqslant C\, \|f\|_2^2 \leqslant C\, \|f\|_{C_\alpha}^2 < \infty.$$

Therefore it suffices to show that

$$\sup_{0<y\leqslant 1} \int_{-\infty}^{\infty} |F(x, y)|^2\, W_\alpha(x, y)\, dx \leqslant C\, \|f\|_{C_\alpha}^2.$$

However,

$$\begin{aligned}\sup_{0<y<1} \int_{-\infty}^{\infty} &|F(x, y)|^2\, W_\alpha(x, y)\, dx \\ &\leqslant \sup_{0<y\leqslant 1} \int_{|x|\leqslant 1} |F(x, y)|^2\, W_\alpha(x, y)\, dx \\ &\quad + \sup_{0<y\leqslant 1} \int_{|x|\geqslant 1} |F(x, y)|^2\, W_\alpha(x, y)\, dx. \qquad (2)\end{aligned}$$

But once again, $|x|>1$ implies that $W_\alpha(x, y) \leqslant C$ and thus the second integral on the right-hand side of (2) is bounded as in (1). So we need only examine the first integral on the right-hand side of (2). Finally, because W_α and the Poisson kernel are both even functions of x, we can reduce our question to showing

$$\sup_{0<y\leqslant 1} \int_0^1 |F(x, y)|^2\, W_\alpha(x, y)\, dx \leqslant C\, \|f\|_{C_\alpha}^2. \qquad (3)$$

Here we can observe that these reductions depended only upon the fact that $W_\alpha(x, y)$ is an even function of x and is uniformly bounded in the complement of $[-1, 1] \times (0, 1]$.

At this stage we proceed to prove (3) by dividing up the regions of integration of the integral in x and the integral defining $F = P[f]$. The iterated integral which we must bound uniformly for $0 < y \leqslant 1$ is

$$I(y) = \int_0^1 \left| y \int_{-\infty}^{\infty} \frac{f(t)\, dt}{(x-t)^2 + y^2} \right|^2 \left(\frac{x^2 + y^2 + 1}{x^2 + y^2} \right)^{\alpha/2} dx. \qquad (4)$$

The guiding principle for our estimation of such integrals is to divide the region of integration at the point where the two summands in the denominator are equal; we then estimate the integrand by retaining the larger summand in each of the resulting regions. Simply stated, for $A>0$ and $B>0$, we use

$$\frac{1}{A+B} \leqslant \begin{cases} 1/A & \text{for } \; A>B \\ 1/B & \text{for } \; B>A. \end{cases}$$

Hence for (4) we divide the outer integral into the regions $X_1 = \{x: 0 \leqslant x \leqslant y\}$ and $X_2 = \{x: y < x \leqslant 1\}$ and the inner integral into the regions $T_1 = \{t: |x-t| \leqslant y\}$ and $T_2 = \{t: |x-t| > y\}$. The resulting integrals are estimated through elementary means and the application of those previously stated properties of the Poisson integral and the maximal function.

X_1, T_1:

$$I(y) \leqslant Cy^2 \int_{X_1} \left(\int_{T_1} \frac{|f(t)|}{y^2} \, dt \right)^2 \frac{1}{y^\alpha} \, dx$$

since $y^2 \geqslant |x-t|^2$, $y^2 \geqslant x^2$, and $x^2 + y^2 + 1 \leqslant 3$. Then by Schwartz' inequality

$$\begin{aligned} I(y) &\leqslant Cy^{-2-\alpha} \int_{X_1} \left(\int_{T_1} \frac{|f(t)|}{|t|^{\alpha/2}} \, |t|^{\alpha/2} \, dt \right)^2 dx \\ &\leqslant Cy^{-2-\alpha} \int_{X_1} \left(\int_{T_1} \frac{|f(t)|^2}{|t|^\alpha} \, dt \right) \left(\int_{T_1} |t|^\alpha \, dt \right) dx. \end{aligned}$$

Here we pause to make the following obvious, but very useful, observation:

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{|t|^\alpha} \, dt \leqslant \int_{-\infty}^{\infty} |f(t)|^2 \left(\frac{t^2+1}{t^2} \right)^{\alpha/2} dt = \|f\|_{C_\alpha}^2 < \infty; \tag{5}$$

i.e., if $f_\alpha(t) = f(t)/|t|^{\alpha/2}$, then $f_\alpha \in L^2(\mathbf{R})$ with $\|f_\alpha\|_2 \leqslant \|f\|_{C_\alpha}$. Thus,

$$I(y) \leqslant C \, \|f\|_{C_\alpha}^2 \, y^{-2-\alpha} \int_{X_1} \left(\int_{T_1} |t|^\alpha \, dt \right) dx.$$

However, $|x-t| \leqslant y$ implies that $|t| \leqslant 2y$ since $x \leqslant y$ so

$$\begin{aligned} I(y) &\leqslant C \, \|f\|_{C_\alpha}^2 \, y^{-2-\alpha} \int_{X_1} (Cy^{\alpha+1}) \, dx \\ &\leqslant C \, \|f\|_{C_\alpha}^2 \, y^{-2-\alpha} y^{\alpha+1} y \\ &= C \, \|f\|_{C_\alpha}^2. \end{aligned}$$

X_1, T_2:

$$I(y) \leqslant Cy^2 \int_{X_1} \left(\int_{T_2} \frac{|f(t)|}{|x-t|^2} \, dt \right)^2 \frac{1}{y^\alpha} \, dx$$

$$= Cy^{2-\alpha} \int_{X_1} \left(\int_{T_2} \frac{|f(t)|}{|t|^{\alpha/2}} \cdot \frac{|t|^{\alpha/2}}{|x-t|^2} \, dt \right)^2 dx$$

$$\leqslant Cy^{2-\alpha} \, \|f\|^2_{C_\alpha} \int_{X_1} \left(\int_{T_2} \frac{|t|^\alpha}{|x-t|^4} \, dt \right) dx$$

by Schwarz' inequality and (5). Note immediately that the inner integral is only defined for $\alpha < 3$ whence the hypothesis in the theorem arises. To evaluate the inner integral for $\alpha < 3$ we make the changes of variable $t = xs$ and then $v = s - 1$:

$$\int_{T_2} \frac{|t|^\alpha}{|x-t|^4} \, dt = \frac{x^{\alpha+1}}{x^4} \int_{|1-s| > y/x} \frac{|s|^\alpha \, ds}{|1-s|^4}$$

$$= x^{\alpha-3} \int_{|v| > y/x} \frac{|v+1|^\alpha}{|v|^4} \, dv$$

$$\leqslant Cx^{\alpha-3} \int_{|v| > y/x} |v|^{\alpha-4} \, dv$$

$$= Cy^{\alpha-3}$$

since $|v| > y/x$ and $y \geqslant x$ imply $|v+1| \leqslant 2\,|v|$. Therefore,

$$I(y) \leqslant C \, \|f\|^2_{C_\alpha} \, y^{2-\alpha} \int_{X_1} y^{\alpha-3} \, dx$$

$$\leqslant C \, \|f\|^2_{C_\alpha} \, y^{2-\alpha} y^{\alpha-3} y = C \, \|f\|^2_{C_\alpha}.$$

X_2, T_1:

$$I(y) \leqslant C \int_{X_2} \left(y \int_{T_1} \frac{|f(t)|}{y^2} \, dt \right)^2 \frac{1}{x^\alpha} \, dx$$

since $y^2 \geqslant |x-t|^2$, $x^2 > y^2$, and $x^2 + y^2 + 1 \leqslant 3$. In this case $|x-t| < y$ and $y < x$ imply $|t| \leqslant x + y < 2x$ and

$$I(y) \leqslant C \int_{X_2} \left(y^{-1} \int_{T_1} \frac{|f(t)|}{|t|^{\alpha/2}} \, |t|^{\alpha/2} \, dt \right)^2 \frac{1}{x^\alpha} \, dx$$

$$\leqslant C \int_{X_2} \left(y^{-1} \int_{T_1} \frac{|f(t)|}{|t|^{\alpha/2}} \, dt \right)^2 dx.$$

Next we observe that the inner integral is precisely twice one of the averages involved in the Hardy–Littlewood maximal function of f_α and therefore by the boundedness of the maximal function and (5)

$$\begin{aligned} I(y) &\leqslant C \int_{X_2} [M(f_\alpha)(x)]^2 \, dx \\ &\leqslant C \, \|f_\alpha\|_2^2 \\ &\leqslant C \, \|f\|_{C_\alpha}^2. \end{aligned}$$

Consider two subsets of T_2: $T_{2B} = \{t: |x-t| > y, |t| > 2x\}$ and $T_{2L} = \{t: |x-t| > y, |t| \leqslant 2x\}$, "B" for big and "L" for little.

X_2, T_{2B}:

$$I(y) \leqslant Cy^2 \int_{X_2} \left(\int_{T_{2B}} \frac{|f(t)|}{|x-t|^2} \, dt \right)^2 \frac{1}{x^\alpha} \, dx.$$

Note that the additional condition $|t| > 2x$ implies that $|x-t| \geqslant ||t|-x| = |t| - x > |t|/2$ and

$$\begin{aligned} I(y) &\leqslant Cy^2 \int_{X_2} \left(\int_{T_{2B}} \frac{|f(t)|}{|t|^2} \, dt \right)^2 \frac{1}{x^\alpha} \, dx \\ &\leqslant Cy^2 \int_{X_2} \left(\int_{T_{2B}} \frac{|f(t)|}{|t|^{\alpha/2}} \cdot \frac{1}{|t|^{2-\alpha/2}} \, dt \right)^2 \frac{1}{x^\alpha} \, dx \\ &\leqslant C \, \|f\|_{C_\alpha}^2 \, y^2 \int_{X_2} \left(\int_{|t| > 2x} |t|^{\alpha - 4} \, dt \right) \frac{1}{x^\alpha} \, dx. \end{aligned}$$

Schwarz' inequality and (5) have been employed here. Again it can be seen that this argument breaks down when $\alpha > 3$. Next we observe that

$$\begin{aligned} I(y) &\leqslant C \, \|f\|_{C_\alpha}^2 \, y^2 \int_{X_2} (x^{\alpha-3} x^{-\alpha}) \, dx \\ &= C \, \|f\|_{C_\alpha}^2 \, y^2 (y^{-2} - 1) \\ &\leqslant C \, \|f\|_{C_\alpha}^2 \end{aligned}$$

since here $y \leqslant 1$.

X_2, T_{2L}:

$$\begin{aligned} I(y) &\leqslant C \int_{X_2} \left(\int_{T_{2L}} \frac{|f(t)| \, dt}{(x-t)^2 + y^2} \right)^2 \frac{1}{x^\alpha} \, dx \\ &\leqslant C \int_{X_2} \left(\int_{T_{2L}} \frac{|f(t)|}{|t|^{\alpha/2}} \cdot \frac{|t|^{\alpha/2}}{(x-t)^2 + y^2} \, dt \right)^2 \frac{1}{x^\alpha} \, dx \\ &\leqslant C \int_{X_2} \left(\int_{T_{2L}} |f_\alpha(t)| \frac{1}{(x-t)^2 + y^2} \, dt \right)^2 dx. \end{aligned}$$

The inner integral is bounded by the Poisson integral of $|f_\alpha|$ thus

$$\begin{aligned} I(y) &\leqslant C \int_{X_2} (P[|f_\alpha|](x, y))^2\, dx \\ &\leqslant C \int_0^1 [M(f_\alpha)(x)]^2\, dx \\ &\leqslant C\, \|Mf_\alpha\|_2^2 \\ &\leqslant C\, \|f_\alpha\|_2^2. \end{aligned}$$

This completes the demonstration of (3) and thus proves the theorem. ∎

It should be remarked that the preceding theorem generalizes completely to n dimensions. The proof is pecisely the same using the n-dimensional Poisson integral and maximal functions.

This provides a description of the Poisson integrals of those portions of $L^2(\mathbf{R})$ included in the spaces C_α. However, there are two shortcomings to Theorem 1. The weights $W_\alpha(x, y) = ((x^2 + y^2 + 1)/(x^2 + y^2))^{\alpha/2}$ do not arise in an organic fashion; they are simply modeled after the weights $w_\alpha(t) = ((t^2 + 1)/t^2)^{\alpha/2}$ characterizing the C_α. Moreover, the theorem does not address the case where $\alpha \geqslant 3$. Indeed the theorem is false for $\alpha > 3$; this is illustrated in the following simple example, again considering the characteristic function of the interval $[1, 2]$.

Let $\alpha > 3$. Then $f = \chi_{[1,2]}$ is in C_α. Consider $y < \frac{1}{2}$;

$$\begin{aligned} &\int_{-\infty}^{\infty} |P[f](x, y)|^2 \left(\frac{x^2 + y^2 + 1}{x^2 + y^2}\right)^{\alpha/2} dx \\ &\quad = y^2 \int_{-\infty}^{\infty} \left(\int_1^2 \frac{dt}{(x-t)^2 + y^2}\right)^2 \left(\frac{x^2 + y^2 + 1}{x^2 + y^2}\right)^{\alpha/2} dx \\ &\quad \geqslant Cy^2 \int_0^y \left(\int_1^2 \frac{dt}{(x-t)^2 + y^2}\right)^2 \frac{1}{y^\alpha}\, dx \\ &\quad = Cy^{2-\alpha} \int_0^y \left(\int_1^2 \frac{dt}{(x-t)^2 + y^2}\right)^2 dx, \end{aligned}$$

when $0 \leqslant x \leqslant y$. Now when $t \in \{t\colon |x - t| > y\}$, either $t < x - y < 0$, which does not intersect $[1, 2]$, or $t > x + y$, which contains $[1, 2]$. Also in this region $(x - t)^2 + y^2 \leqslant 2(x - t)^2$ and thus

$$\int_{-\infty}^{\infty} |P[f](x, y)|^2 \left(\frac{x^2+y^2+1}{x^2+y^2}\right)^{\alpha/2} dx$$

$$\geqslant Cy^{2-\alpha} \int_0^y \left(\int_1^2 \frac{dt}{(x-t)^2}\right)^2 dx \geqslant Cy^{2-\alpha} \int_0^y \left(\int_1^2 \frac{dt}{t^2}\right)^2 dx$$

since $(x-t)^2 < t^2$ here. Therefore,

$$\int_{-\infty}^{\infty} |P[f](x, y)|^2 \left(\frac{x^2+y^2+1}{x^2+y^2}\right)^{\alpha/2} dx \geqslant Cy^{3-\alpha} \qquad C \neq 0,$$

which is unbounded for y approaching zero.

This shortfall and the other mentioned above are both remedied through a somewhat different approach. We would like to find a weight against which the Poisson integrals of certain classes of functions are square-integrable. Since the Poisson integral of L^2 functions are well-behaved and since $f \in C_\alpha$ implies that $f\sqrt{w_\alpha} \in L^2$, it might be reasonable to consider using $P[w_\alpha]$ as the requisite weight. However, for $\alpha > 1$, w_α is not even locally integrable and thus not in any L^p space. But w_α is bounded below, hence its reciprocal is bounded. This is the observation we exploit.

THEOREM 2. *If* $W_\alpha(x, y) = (P[w_\alpha^{-1}](x, y))^{-1}$, *then* $D_\alpha = \tilde{D}_\alpha$ *for all* $\alpha \geqslant 0$. *Moreover, if* $F \in D_\alpha$ *where* $F = P[f]$, *then*

$$\|f\|_{C_\alpha} \leqslant \|F\|_{\tilde{D}_\alpha} \leqslant C\, \|f\|_{C_\alpha}.$$

Proof. The inclusion of $\tilde{D}_\alpha$ in D_α follows exactly as in the proof of Theorem 1: Since w_α is bounded below, so is W_α due to the domination of the Poisson integral by the maximal function. This again implies the existence of an $f \in L^2(\mathbf{R})$ such that $F = P[f]$. The only other property used was that $\lim_{y \to 0} W_\alpha(x, y) = w_\alpha(x)$. Since $w_\alpha^{-1}(t) = (t^2/(t^2+1))^{\alpha/2}$ is continuous, this is an immediate consequence of the convergence of the Poisson integral to its boundary values as $y \to 0$.

To reduce the converse, proving that D_α is a subspace of $\tilde{D}_\alpha$, to consideration of inequality (3), we need to assert two properties of the W_α's now under discussion. First, $W_\alpha(x, y)$ is clearly an even function of x, as it is basically the Poisson integral of an even function. Next, it remains to demonstrate that $W_\alpha(x, y)$ is uniformly bounded on the complement of $[0, 1] \times (0, 1]$. This is accomplished by showing that $P[w_\alpha^{-1}](x, y)$ is bounded below, away from zero. We do this in two cases; the techniques employed are strongly reminiscent of those we have used earlier.

Case 1. $x > 1$.

$$
\begin{aligned}
P[w_\alpha^{-1}](x, y) &= \frac{y}{\pi} \int_{|x-t|>y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2+y^2} \\
&\quad + \frac{y}{\pi} \int_{|x-t|\leqslant y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2+y^2} \\
&\geqslant \frac{y}{\pi} \int_{|x-t|\leqslant y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{2y^2} \\
&\geqslant \frac{1}{2\pi y} \int_{\substack{|x-t|\leqslant y \\ |t|>1}} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} dt \\
&\geqslant \frac{1}{2\pi 2^{\alpha/2}} \frac{1}{y} \int_{\substack{|x-t|\leqslant y \\ |t|>1}} dt
\end{aligned}
$$

since

$$
\left(\frac{t^2}{t^2+1}\right)^{\alpha/2} = \left(\frac{1}{1+t^{-2}}\right)^{\alpha/2} \geqslant 2^{-\alpha/2}
$$

here. The last integral is

$$
(2^{\alpha/2+1}\pi y)^{-1} \int_{x-y}^{x+y} dt
$$

subject to $|t| > 1$. This gives rise to three possibilities.

$$
\begin{aligned}
P[w_\alpha^{-1}](x, y) &\geqslant (2^{\alpha/2+1}\pi y)^{-1} \int_{x-y}^{x+y} dt \\
&= (2^{\alpha/2}\pi)^{-1} > 0 \qquad \text{if} \quad x - y > 1. \\
P[w_\alpha^{-1}](x, y) &\geqslant (2^{\alpha/2+1}\pi y)^{-1} \int_{1}^{x+y} dt \\
&= (2^{\alpha/2+1}\pi)^{-1} \left(1 + \frac{x-1}{y}\right) \\
&> (2^{\alpha/2+1}\pi)^{-1} > 0 \qquad \text{if} \quad -1 \leqslant x - y \leqslant 1. \\
P[w_\alpha^{-1}](x, y) &\geqslant (2^{\alpha/2+1}\pi y)^{-1} \left[\int_{x-y}^{-1} dt + \int_{1}^{x+y} dt\right] \\
&\geqslant (2^{\alpha/2+1}\pi y)^{-1} \int_{1}^{x+y} dt \\
&= (2^{\alpha/2+1}\pi)^{-1} \left(1 + \frac{x-1}{y}\right) \\
&> (2^{\alpha/2+1}\pi)^{-1} > 0, \qquad \text{if} \quad x - y < -1.
\end{aligned}
$$

Case 2. $0 \leqslant x \leqslant 1$, $y > 1$.

$$\begin{aligned} P[w_\alpha^{-1}](x, y) &= \frac{y}{\pi} \int_{|x-t|>y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2+y^2} \\ &\quad + \frac{y}{\pi} \int_{|x-t|\leqslant y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2+y^2} \\ &\geqslant \frac{y}{\pi} \int_{|x-t|>y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{2(x-t)^2} \\ &\geqslant \frac{y}{2\pi} \int_{\substack{|x-t|>y \\ |t|>2}} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2} \\ &\geqslant (2^{\alpha/2+1}\pi)^{-1}\, y \int_{\substack{|x-t|>y \\ |t|>2}} \frac{dt}{(x-t)^2} \end{aligned}$$

since $(t^2/(t^2+1))^{\alpha/2} > 2^{-\alpha/2}$ again. Let $\beta = (2^{\alpha/2+1}\pi)^{-1} > 0$. The last integral leads to the consideration of four integrals depending upon the sizes of $x + y$ and $x - y$; we can estimate each successfully.

$$\beta y \int_{x+y}^{\infty} \frac{dt}{(x-t)^2} = \beta y \left(\frac{1}{y}\right) = \beta > 0$$

$$\beta y \int_{2}^{\infty} \frac{dt}{(x-t)^2} = \beta y \left(\frac{1}{2-x}\right) > \beta/2 > 0 \qquad \text{since} \quad x \geqslant 0 \text{ and } y > 1$$

$$\beta y \int_{-\infty}^{x-y} \frac{dt}{(x-t)^2} = \beta y \left(\frac{1}{y}\right) = \beta > 0$$

$$\beta y \int_{-\infty}^{-2} \frac{dt}{(x-t)^2} = \beta y \left(\frac{1}{x+2}\right) > \beta/3 > 0 \qquad \text{since} \quad x \leqslant 1 \text{ and } y > 1.$$

This demonstrates the boundedness of $W_\alpha = (P[w_\alpha^{-1}])^{-1}$ in the complement of $[0, 1] \times (0, 1]$, and hence we are left with proving that

$$\sup_{0<y\leqslant 1} \int_0^1 |F(x, y)|^2\, W_\alpha(x, y)\, dx \leqslant C\, \|f\|_{C_\alpha}^2$$

or

$$\sup_{0<y\leqslant 1} \int_0^1 |P[f](x, y)|^2\, (P[w_\alpha^{-1}](x, y))^{-1}\, dx \leqslant C\, \|f\|_{C_\alpha}^2. \tag{6}$$

In the region $[0, 1] \times (0, 1]$ we consider two situations: $0 \leqslant \alpha \leqslant 3$ and

$\alpha > 3$. The first task is to determine the growth of $W_\alpha = (P[w_\alpha^{-1}])^{-1}$ for these two cases. Again we consider $W_\alpha^{-1} = P[w_\alpha^{-1}]$, and in general we see

$$\begin{aligned} P[w_\alpha^{-1}](x, y) &= \frac{y}{\pi} \int_{|x-t| \leqslant y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2+y^2} \\ &\quad + \frac{y}{\pi} \int_{|x-t| > y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2+y^2} \\ &= S_1 + S_2. \end{aligned}$$

Observe that for $0 \leqslant x \leqslant 1$ and $0 < y \leqslant 1$,

$$\{t \colon |x-t| \leqslant y\} \subset \{t \colon |t| \leqslant 2\}, \tag{7a}$$

and

$$\{t \colon |x-t| > y\} \supset \{t \colon |t| > 2\}. \tag{7b}$$

This groundwork yields the estimates

$$\begin{aligned} S_1 &\geqslant \frac{y}{\pi} \int_{|x-t| \leqslant y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{2y^2} \geqslant \frac{1}{2\pi y} \int_{|x-t| \leqslant y} \frac{|t|^\alpha}{5^{\alpha/2}}\, dt \\ &= (2\pi 5^{\alpha/2} y)^{-1} \int_{|x-t| \leqslant y} |t|^\alpha\, dt \end{aligned}$$

by (7a) and

$$\begin{aligned} S_2 &\geqslant \frac{y}{\pi} \int_{|x-t| > y} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{2(x-t)^2} \\ &\geqslant \frac{y}{2\pi} \int_{|t| > 2} \left(\frac{t^2}{t^2+1}\right)^{\alpha/2} \frac{dt}{(x-t)^2} \geqslant (2^{\alpha/2+1}\pi)^{-1}\, y \int_{|t| > 2} \frac{dt}{(x-t)^2} \end{aligned}$$

by (7b). Since $0 \leqslant x \leqslant 1$ and $|t| > 2$, then

$$|t-x| \leqslant |t| + |x| \leqslant |t| + 1;$$

so we also have

$$S_2 \geqslant Cy \int_{|t| > 2} \frac{dt}{(|t|+1)^2} = Cy.$$

These estimates for S_1 and S_2 lead to the following estimates on W_α:

$\alpha \geqslant 3$:

$$S_2 \geqslant Cy$$

$$S_1 \geqslant \frac{C}{y} \int_{|x-t| \leqslant y} |t|^\alpha \, dt \geqslant \frac{C}{y} \int_x^{x+y} t^\alpha \, dt \geqslant \frac{C}{y} (yx^\alpha) = Cx^\alpha$$

by the mean value theorem for integrals. Therefore, in the region $[0, 1] \times (0, 1]$ and for these α, we have

$$P[w_\alpha^{-1}](x, y) \geqslant C(x^\alpha + y).$$

Hence

$$W_\alpha(x, y) = (P[w_\alpha^{-1}](x, y))^{-1} \leqslant \frac{C}{x^\alpha + y}. \tag{8}$$

$0 \leqslant \alpha < 3$: If $0 < y \leqslant x \leqslant 1$, then we can estimate $S_1 \geqslant Cx^\alpha$ as above. For $0 \leqslant x < y \leqslant 1$,

$$S_1 \geqslant \frac{C}{y} \int_{x-y}^{x+y} |t|^\alpha \, dt \geqslant \frac{C}{y} \int_0^{x+y} t^\alpha \, dt = \frac{C}{y} (x+y)^{\alpha+1} \geqslant Cy^\alpha.$$

So here we have

$$P[w_\alpha^{-1}](x+y) \geqslant C(x^\alpha + y^\alpha),$$

and that implies

$$W_\alpha(x, y) = (P[w_\alpha^{-1}](x, y))^{-1} \leqslant \frac{C}{x^\alpha + y^\alpha}. \tag{9}$$

It is pertinent to observe here that estimate (8) holds equally well for $1 < \alpha < 3$, and, therefore, the weight currently under discussion is actually smaller than the estimate (9) employed in this range. However, it is no more difficult to establish inequality (6) for $1 < \alpha < 3$ using the weaker estimate (9), and it is a somewhat stronger result. As the last example illustrated, estimate (9) is not sufficient for $\alpha > 3$.

Let us recall where we are now. Through a general discussion of $W_\alpha = (P[w_\alpha^{-1}])^{-1}$ and following the proof of Theorem 1 we proved that $\tilde{D}_\alpha$ is contained in D_α and reduced the demonstration of the inclusion of D_α in $\tilde{D}_\alpha$ to proving (6), i.e., consideration of $(x, y) \in [0, 1] \times (0, 1]$. In this region we inspected $(P[w_\alpha^{-1}])^{-1}$ separately in the cases $\alpha \geqslant 3$ and $0 \leqslant \alpha < 3$. It is left now to deduce the inequality (6) for each of those cases; we begin with the second.

$0 \leqslant \alpha < 3$: The goal is to demonstrate the following inequality which results from incorporating (9) into inequality (6):

$$\sup_{0<y\leqslant 1} \int_0^1 |P[f](x,y)|^2 \left(\frac{1}{x^\alpha + y^\alpha}\right) dx \leqslant C \|f\|^2_{C_\alpha}. \tag{10}$$

Our program is again imitative of the proof of Theorem 1. We split both the integral in x and the integral defining $P[f]$ at points where the summands in their respective denominators are equal. Note that for the x-integral in (10) this amounts to considering $X_1 = \{x: 0 \leqslant x \leqslant y\}$ and $X_2 = \{x: y < x \leqslant 1\}$. In the former we would proceed to estimate $(x^\alpha + y^\alpha)^{-1}$ by $y^{-\alpha}$ and in the latter by $x^{-\alpha}$. However, observe that this leads to precisely the estimations made in the proof of Theorem 1, and since $0 \leqslant \alpha < 3$ here, all of those earlier deductions are valid. Therefore inequality (10) is proved and the proof of the present theorem is complete in this case.

$\alpha > 3$: Here we wish to prove that

$$\sup_{0<y\leqslant 1} \int_0^1 |P[f](x,y)|^2 \left(\frac{1}{x^\alpha + y}\right) dx \leqslant C \|f\|^2_{C_\alpha}, \tag{11}$$

or, in other words, we wish to show that

$$J(y) = \int_0^1 \left(y \int_{-\infty}^{\infty} \frac{|f(t)|}{(x-t)^2 + y^2}\, dt\right)^2 \left(\frac{1}{x^\alpha + y}\right) dx$$

is uniformly bounded by a multiple of $\|f\|^2_{C_\alpha}$ for $0 < y \leqslant 1$. Again the plan is to divide up the two integrals involved and, using various methods, to estimate their sizes. Many of the pieces will look and be familiar. In the interest of completeness, all of the details are presented; in the interest of compactness, they are somewhat abridged. Bear in mind that we have restricted our purview to $0 \leqslant x \leqslant 1$, $0 < y \leqslant 1$, and $\alpha \geqslant 3$.

The outer integral will be split into two pieces: $X_1 = \{x: x^\alpha \leqslant y\}$ and $X_2 = \{x: x^\alpha > y\}$. Within each of these regions, the inner integral will be divided into four pieces:

$T_1 = \{t: |x-t| \leqslant y\}$,
$T_{2B} = \{t: |x-t| > y, |t| > 2\}$,
$T_{2M} = \{t: |x-t| > y, 2x < |t| \leqslant 2\}$, and
$T_{2L} = \{t: |x-t| > y, |t| \leqslant 2x\}$.

It will also be useful to consider the following supersets:

$T_B = \{t: |t| > 2\}$,
$T_M = \{t: 2x < |t| \leqslant 2\}$, and
$T_L = \{t: |t| \leqslant 2x\}$.

X_1, T_1:

$$J(y)=\int_{X_1}\left(y\int_{T_1}\frac{|f(t)|}{y^2}\,dt\right)^2\frac{1}{y}\,dx$$

$$=\int_{X_1}\left(\frac{1}{y}\int_{T_1}\frac{|f(t)|}{|t|^{\alpha/2}}\,|t|^{\alpha/2}\,dt\right)^2\frac{1}{y}\,dx.$$

Note that $|x-t|\leqslant y$ implies that $x-y\leqslant t\leqslant x+y$ which means that $|t|<x+y<y^{1/\alpha}+y<2y^{1/\alpha}$ since $\alpha\geqslant 3$ and thus $|t|^\alpha<Cy$. Therefore,

$$J(y)\leqslant C\int_{X_1}\left(\frac{1}{y}\int_{T_1}\frac{|f(t)|}{|t|^{\alpha/2}}\,dt\right)^2 dx$$

$$\leqslant C\int_{X_1}[M(f_\alpha)(x)]^2\,dx$$

$$\leqslant C\,\|f_\alpha\|_2^2\leqslant C\,\|f\|_{C_\alpha}^2.$$

Recall that $f_\alpha(t)=f(t)/|t|^{\alpha/2}$, and $f\in C_\alpha$ implies $f_\alpha\in L^2(\mathbf{R})$ with $\|f_\alpha\|_2\leqslant\|f\|_{C_\alpha}$. We have also called upon the boundedness of the Hardy–Littlewood maximal function on $L^2(\mathbf{R})$.

$X_1, T_{2\mathrm{B}}$:

$$J(y)\leqslant\int_{X_1}\left(y\int_{T_{2\mathrm{B}}}\frac{|f(t)|}{(x-t)^2}\,dt\right)^2\frac{1}{y}\,dx$$

$$\leqslant y\int_0^1\left(\int_{T_\mathrm{B}}\frac{|f(t)|}{(x-t)^2}\,dt\right)^2 dx$$

$$\leqslant y\,\|f\|_2^2\int_0^1\left(\int_{T_\mathrm{B}}\frac{dt}{(x-t)^4}\right)dx$$

$$\leqslant\|f\|_{C_\alpha}^2\int_0^1\left(\int_{T_\mathrm{B}}\frac{dt}{(|t|-1)^4}\right)dx$$

$$\leqslant C\,\|f\|_{C_\alpha}^2.$$

Here we used the immediate observations that

$$|t-x|\geqslant|t|-x>|t|-1$$

and

$$\int|f(t)|^2\,dt\leqslant\int|f(t)|^2\left(\frac{t^2+1}{t^2}\right)^{\alpha/2}dt,$$

and we remember that $0<y\leqslant 1$.

X_1, T_{2M}:

$$J(y) \leqslant \int_{X_1} \left(y \int_{T_{2M}} \frac{|f(t)|}{(x-t)^2} \, dt \right)^2 \frac{1}{y} \, dx$$

$$\leqslant Cy \int_{X_1} \left(\int_{T_M} \frac{|f(t)|}{|t|^{\alpha/2}} \cdot \frac{|t|^{\alpha/2}}{t^2} \, dt \right)^2 dx$$

since $|t-x| > |t| - x > |t|/2$. Therefore,

$$J(y) \leqslant C \, \|f\|_{C_\alpha}^2 \, y \int_{X_1} \left(\int_{T_M} |t|^{\alpha-4} \, dt \right) dx$$

$$\leqslant \begin{cases} C \, \|f\|_{C_\alpha}^2 \, y^{1+1/\alpha} \leqslant C \, \|f\|_{C_\alpha}^2 & \alpha > 3 \\ C \, \|f\|_{C_\alpha}^2 \, y \int_0^1 (\log 2 - \log 2x) \, dx \leqslant C \, \|f\|_{C_\alpha}^2 & \alpha = 3 \end{cases}$$

recalling that $0 < y \leqslant 1$.

X_1, T_{2L}:

$$J(y) \leqslant \int_{X_1} \left(y \int_{T_{2L}} \frac{|f(t)|}{(x-t)^2 + y^2} \, dt \right)^2 \frac{1}{y} \, dx$$

$$\leqslant \int_{X_1} \left(y \int_{T_L} \frac{|f(t)|}{|t|^{\alpha/2}} \cdot \frac{|t|^{\alpha/2}}{(x-t)^2 + y^2} \, dt \right)^2 \frac{1}{y} \, dx$$

$$\leqslant C \int_0^1 \left(y \int_{T_L} |f_\alpha(t)| \frac{dt}{(x-t)^2 + y^2} \right)^2 dx$$

since $|t| \leqslant 2x$ implies that $|t|^\alpha < Cx^\alpha \leqslant Cy$, and thus

$$J(y) \leqslant C \int_0^1 (P[|f_\alpha|](x, y))^2 \, dx$$

$$\leqslant C \int_0^1 [M(f_\alpha)(x)]^2 \, dx$$

$$\leqslant C \, \|f_\alpha\|_2^2 \leqslant C \, \|f\|_{C_\alpha}^2.$$

X_2, T_1:

$$J(y) \leqslant \int_{X_2} \left(y \int_{T_1} \frac{|f(t)|}{y^2} \, dt \right)^2 \frac{1}{x^\alpha} \, dx$$

$$= \int_{X_2} \left(\frac{1}{y} \int_{T_1} \frac{|f(t)|}{|t|^{\alpha/2}} \, |t|^{\alpha/2} \, dt \right)^2 \frac{1}{x^\alpha} \, dx$$

$$\leqslant C \int_{X_2} \left(\frac{1}{y} \int_{T_1} |f_\alpha(t)| \, dt \right)^2 dx$$

since $|x-t| \leqslant y$ implies $|t| \leqslant x+y \leqslant x+x^\alpha \leqslant 2x$ since $\alpha \geqslant 3$, thus $|t|^\alpha \leqslant Cx^\alpha$, and therefore,

$$\begin{aligned} J(y) &\leqslant C \int_0^1 [M(f_\alpha)(x)]^2\, dx \\ &\leqslant C \|f_\alpha\|_2^2 \leqslant C \|f\|_{C_\alpha}^2. \end{aligned}$$

X_2, $T_{2\mathrm{B}}$:

$$\begin{aligned} J(y) &\leqslant \int_{X_2} \left(y \int_{T_{2\mathrm{B}}} \frac{|f(t)|}{(x-t)^2}\, dt \right)^2 \frac{1}{x^\alpha}\, dx \\ &\leqslant y \int_{X_2} \left(\int_{T_\mathrm{B}} \frac{|f(t)|}{(x-t)^2}\, dt \right)^2 dx \end{aligned}$$

since $x^{-\alpha} < y^{-1}$. Hence, using Schwarz' inequality, the fact that $|x-t| \geqslant |t|-x \geqslant |t|-1$, $\|f\|_2 \leqslant \|f\|_{C_\alpha}$, and $0<y\leqslant 1$,

$$\begin{aligned} J(y) &\leqslant y \|f\|_2^2 \int_0^1 \left(\int_{T_\mathrm{B}} \frac{1}{(|t|-1)^4}\, dt \right) dx \\ &\leqslant C \|f\|_{C_\alpha}^2. \end{aligned}$$

X_2, $T_{2\mathrm{M}}$:

$$\begin{aligned} J(y) &\leqslant \int_{X_2} \left(y \int_{T_{2\mathrm{M}}} \frac{|f(t)|}{(x-t)^2}\, dt \right)^2 \frac{1}{x^\alpha}\, dx \\ &\leqslant Cy^2 \int_{X_2} \left(\int_{T_\mathrm{M}} \frac{|f(t)|}{|t|^{\alpha/2}} \cdot \frac{|t|^{\alpha/2}}{t^2}\, dt \right)^2 \frac{1}{x^\alpha}\, dx \end{aligned}$$

again, since $|x-t| \geqslant |t|-x > |t|/2$. By Schwarz' inequality,

$$\begin{aligned} J(y) &\leqslant C \|f\|_{C_\alpha}^2 y^2 \int_{X_2} \left(\int_{T_\mathrm{M}} |t|^{\alpha-4}\, dt \right) \frac{1}{x^\alpha}\, dx \\ &\leqslant \begin{cases} C \|f\|_{C_\alpha}^2 y \int_0^1 dx \leqslant \|f\|_{C_\alpha}^2 & \alpha > 3, \text{ since } x^{-\alpha} < y^{-1} \\ C \|f\|_{C_\alpha}^2 y^2 \int_{x^3>y} \left(\int_{|t|\leqslant 2} x^{-1}\, dt \right) \frac{1}{x^3}\, dx & \alpha = 3 \end{cases} \\ &= C \|f\|_{C_\alpha}^2 y^2 (y^{-1/3} - 1) \\ &\leqslant C \|f\|_{C_\alpha}^2, \qquad \text{since} \quad 0 < y \leqslant 1. \end{aligned}$$

X_2, T_{2L}:

$$J(y) \leqslant \int_{X_2} \left(y \int_{T_{2L}} \frac{|f(t)|}{|t|^{\alpha/2}} \cdot \frac{|t|^{\alpha/2}}{(x-t)^2+y^2}\,dt \right)^2 \frac{1}{x^\alpha}\,dx$$
$$\leqslant C \int_{X_2} (P[|f_\alpha|](x, y))^2\,dx$$

since $|t| \leqslant 2x$, and, therefore,

$$J(y) \leqslant C \int_0^1 [M(f_\alpha)(x)]^2\,dx$$
$$\leqslant C \, \|f_\alpha\|_2^2 \leqslant C \, \|f\|_{C_\alpha}^2$$

employing the usual tools and our favorite estimate on f_α.

Inequality (11) is established, this case is finished, and the theorem is proved. ▮

The characterization of the Poisson integrals of all those square-integrable functions which vanish sufficiently rapidly at the origin to remain square-integrable against the singular weights $w_\alpha(t)$ which behave like $|t|^{-\alpha}$ is complete. The corresponding weight for the Poisson integrals arises as the reciprocal of the Poisson integral of the reciprocal of w_α. In the case of $0 \leqslant \alpha < 3$ an explicit weight is found by mimicking w_α.

References

1. A. P. Calderon and A. Zygmund, On the existence of certain singular integrals, *Acta Math.* **88** (1952), 85–139.
2. C. P. Slavin, Properties of power-series coefficients of $H^2(\pi_+)$ functions, *J. Math. Anal. Appl.* **142** (1989), 488–507.
3. K. T. Smith, A generalization of an inequality of hardy and Littlewood, *Canad. J. Math.* **8** (1956), 157–170.
4. E. M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, NJ, 1970.
5. E. C. Titchmarsh, "Introduction to the Theory of Fourier Integrals," 2nd ed., Oxford Univ. Press, London, 1948.